

KSGNS TYPE CONSTRUCTIONS FOR α -COMPLETELY POSITIVE MAPS ON KREIN C^* -MODULES

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ABSTRACT. In this paper, we investigate Φ -maps associated to a certain type of α -completely positive maps. We then prove a KSGNS (Kasparov–Stinespring–Gel’fand–Naimark–Segal) type theorem for α -completely positive maps on Krein C^* -modules and show that the minimal KSGNS construction is unique up to unitary equivalence. We also establish a covariant version of the KSGNS type theorem for a covariant α -completely positive map and study the structure of minimal covariant KSGNS constructions.

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper by a map (an operator) we mean a bounded linear one.

1.1. Completely Positive maps. Completely positive maps can be regarded as extensions of states, representations and conditional expectations. A (not necessarily linear) map between C^* -algebras, which sends positive elements to positive elements is called positive. By a completely positive map we mean a map $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ between C^* -algebras with the property that for each n , φ is n -positive, in the sense that, the map φ_n from the C^* -algebra $\mathcal{M}_n(\mathcal{A})$ of all $n \times n$ matrices with entries in \mathcal{A} into $\mathcal{M}_n(\mathcal{B})$ defined by $\varphi([a_{ij}]) = [\varphi(a_{ij})]$ is positive. For example, any positive map $\varphi : C(X) \rightarrow C(Y)$ between unital C^* -algebras is completely positive. There are positive maps which are not completely positive, for instance, the map $\varphi : \mathcal{M}_3(\mathbb{C}) \rightarrow \mathcal{M}_3(\mathbb{C})$ defined by $\varphi(X) = 2\text{tr}(X)I - X$ is 2-positive but not 3-positive.

In quantum information theory, a quantum operation is defined as a certain completely positive linear map and plays an essential role in describing the transformations, which a quantum mechanical system may undergo. The structure of completely positive maps, the description of the order relation on their set, the characterization of pure maps and extremal maps according to their structure are significant in understanding a lot of problems, see [11].

The Stinespring theorem for C^* -algebras is a significant generalization of the Gel’fand–Naimark–Segal (GNS) construction to operator-valued maps. The GNS construction gives a correspondence between cyclic $*$ -representations of a C^* -algebra and its certain linear functionals and used to establish the celebrated Gel’fand–Naimark–Segal theorem, which characterize C^* -algebras as algebras of Hilbert space operators, see [8]. The classical version of Stinespring’s theorem states that a map T from a unital C^* -algebra \mathcal{A} into the C^* -algebra $\mathcal{L}(\mathcal{H})$ of all (bounded linear) operators is completely

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positive if and only if it is of the form $T(a) = V^* \pi(a) V$ ($a \in \mathcal{A}$), where $\pi : \mathcal{A} \rightarrow \mathcal{L}(K)$ is a $*$ -representation of \mathcal{A} and $V : \mathcal{H} \rightarrow \mathcal{K}$ is a map. Kasparov [19] extended the Stinespring theorem for completely positive linear maps from a C^* -algebra \mathcal{A} to the C^* -algebra of all adjointable operators on the Hilbert C^* -module $\mathcal{H}_{\mathcal{B}}$ over C^* -algebra \mathcal{B} . He showed that a completely positive linear map φ from \mathcal{A} to $\mathcal{H}_{\mathcal{B}}$ is of the form $\varphi(a) = V^* \pi(a) V$ ($a \in \mathcal{A}$), where $\pi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ is a $*$ -representation of \mathcal{A} on a Hilbert C^* -module \mathcal{X} and $V : \mathcal{H}_{\mathcal{B}} \rightarrow \mathcal{X}$ is a bounded linear map. Bhat et al. [5] extended the Stinespring theorem for completely positive φ -maps on Hilbert C^* -modules. They provided a Stinespring construction associated to a completely positive φ -map Φ on a Hilbert C^* -module X in terms of the Stinespring construction associated to the underlying completely positive map φ . A covariant version of this construction can be found in [18]. Several generalizations of Stinespring theorem are given by mathematicians; see [1] and references therein.

On the other hand, the notion of locality in the Wightman formulation of gauge quantum field theory [24, 17] conflicts with the notion of positivity. To avoid this, Jakobczyk and Strocchi [17] introduced the concept of α -positivity. A typical example of an α -positive map is as follows, cf. [3].

Example. Let $\mathcal{B} \subseteq \mathcal{A}$ be unital C^* -algebras and $P : \mathcal{A} \rightarrow \mathcal{B}$ be a conditional expectation, i.e. a unital $*$ -map satisfying $P(b_1 a b_2) = b_1 P(a) b_2$ ($a \in \mathcal{A}, b_1, b_2 \in \mathcal{B}$). Let ρ be an P -functional, i.e. a Hermitian functional on \mathcal{A} such that $\rho(P(a)) = \rho(a)$ and $2\rho(P(a)^* P(a)) \geq \rho(a^* a)$ for all $a \in \mathcal{A}$. Then $\alpha(a) := 2P(a) - a$ is a linear involution, i.e. $\alpha^2(a) = a$ for all $a \in \mathcal{A}$ and satisfies $\rho(\alpha(a_1) \alpha(a_2)) = \rho(a_1 a_2)$ and $\rho(\alpha(a)^* a) \geq 0$ for all $a_1, a_2, a \in \mathcal{A}$.

Motivated by the notion of α -positivity and P -functionals (see [16, 3]), Heo et al. [12] introduced the notion of α -completely positive map between C^* -algebras and provided a Kasparov–Stinespring–Gelfand–Naimark–Segal (KSGNS) type construction for α -completely positive maps. Here, positivity is inherent in Hermitian maps in terms of the map α . The α -complete positivity provides a positive definite inner product associated to the indefinite one, and the interplay between these two is indeed the characteristic feature of Krein spaces among all indefinite inner product spaces.

1.2. Hilbert C^* -modules. Hilbert C^* -modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, takes its values in a C^* -algebra. Although Hilbert C^* -modules behave like Hilbert spaces in some ways, some fundamental Hilbert space properties like Pythagoras equality, the adjointability of operators and the decomposition into orthogonal complements do not hold in general. An inner product C^* -module over a C^* -algebra \mathcal{A} is a complex linear space \mathcal{X} which is a right \mathcal{A} -module with a compatible scalar multiplication (i.e., $\gamma(xa) = (\gamma x)a = x(\gamma a)$ for all $x \in \mathcal{X}, a \in \mathcal{A}, \gamma \in \mathbb{C}$) and equipped with an \mathcal{A} -valued inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{A}$ satisfying

- (i) $\langle x, \gamma y + \mu z \rangle = \gamma \langle x, y \rangle + \mu \langle x, z \rangle$,
- (ii) $\langle x, ya \rangle = \langle x, y \rangle a$,
- (iii) $\langle x, y \rangle^* = \langle y, x \rangle$,
- (iv) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,

for all $x, y, z \in \mathcal{X}, a \in \mathcal{A}, \gamma, \mu \in \mathbb{C}$. It is easy to observe that $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ defines

a norm on \mathcal{X} , where the later norm is that of \mathcal{A} . If \mathcal{X} with respect to this norm is complete, then it is called a *Hilbert \mathcal{A} -module*, or a *Hilbert C^* -module* over \mathcal{A} . Complex Hilbert spaces are (left) Hilbert \mathbb{C} -modules. Any C^* -algebra \mathcal{A} can be regarded as a Hilbert C^* -module over itself via $\langle a, b \rangle := a^*b$. The Hilbert \mathcal{A} -module \mathcal{X} is called full if the ideal generated by $\{\langle x, y \rangle : x, y \in \mathcal{X}\}$ is dense in \mathcal{A} . A map T from a Hilbert \mathcal{A} -module \mathcal{X} into another Hilbert \mathcal{A} -module \mathcal{Y} is adjointable if there is a map $T^* : \mathcal{Y} \rightarrow \mathcal{X}$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. It is easy to show that any adjointable map T is a bounded \mathcal{A} -module one. We denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of all adjointable module morphisms from \mathcal{X} to \mathcal{Y} . In the case that $\mathcal{X} = \mathcal{Y}$, it is denoted by $\mathcal{L}(\mathcal{X})$, which is a C^* -algebra. The reader is referred to [21] for basic notions related to Hilbert C^* -modules.

Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a linear map. A map Φ from a Hilbert \mathcal{A} -module \mathcal{X} into another Hilbert \mathcal{B} -module \mathcal{Y} is called a φ -map if $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$ for all $x, y \in \mathcal{X}$. A φ -map Φ is completely positive if φ is completely positive.

1.3. Krein spaces. A Krein space, as an indefinite generalization of a Hilbert space, is a vector space equipped with a symmetric or Hermitian bilinear form $[\cdot, \cdot]$ in such a way that $[x, x]$ can be positive, negative or zero, [6, 4] for more information. This notion was first defined by Ginzburg [10]. Indeed lack of positivity in some models in quantum field theories made theoretical physicists to consider indefinite structures. Since then many mathematicians have investigated them. Recently Heo et al. [12, 15] studied Krein C^* -modules and covariant representations on Krein C^* -modules. A treatment of operator convex functions is presented in [22].

Motivated by the classical theory of Krein spaces, we can introduce a parallel theory to the setting of Hilbert C^* -modules. For an exposition on the subject see [2].

Definition 1.1. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert \mathcal{A} -module over a C^* -algebra \mathcal{A} and let J be a fundamental symmetry on \mathcal{H} , i.e., $J = J^* = J^{-1}$. Then one can define an indefinite \mathcal{A} -valued inner product by

$$[x, y] := \langle Jx, y \rangle \quad (x, y \in \mathcal{H}).$$

In this case (\mathcal{H}, J) is called a Krein \mathcal{A} -module.

For a Krein \mathcal{A} -module (\mathcal{H}, J) , if $\mathcal{H}_+, \mathcal{H}_-$ are the ranges of projections $P_+ = (I + J)/2$, $P_- = (I - J)/2$, where I denote the identity operator, then one obtains the orthogonal direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, $J = P_+ - P_-$, $[x, x] = |P_+x|^2 - |P_-x|^2$ and $\langle x, x \rangle = |P_+x|^2 + |P_-x|^2$. If $\mathcal{A} = \mathbb{C}$, then we reach the classical definition of a Krein space and its fundamental structure. Obviously, if $J = I$, then the theory reduces to the theory of Hilbert spaces.

Definition 1.2. Let \mathcal{A} be a C^* -algebra and α be a $*$ -automorphism such that α^2 is the identity operator. Evidently if \mathcal{A} is unital, then $\alpha(1) = 1$. One can define an indefinite involution $x^\# = \alpha(x^*)$ on \mathcal{A} . Then (\mathcal{A}, α) is called a Krein C^* -algebra. Thus $\|\alpha(x^\#)x\| = \|x\|^2$ for all $x \in \mathcal{A}$. It is easy to see that $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$, where $\mathcal{A}_+ = \{x \in \mathcal{A} | \alpha(x) = x\}$ and $\mathcal{A}_- = \{x \in \mathcal{A} | \alpha(x) = -x\}$. In addition, \mathcal{A}_+ is a C^* -algebra and (\mathcal{A}_-, α) is a Krein C^* -module over \mathcal{A}_+ with $[x, y] = x^\#y = \alpha(x^*)y = -x^*y$ for all $x, y \in \mathcal{A}_-$.

Example. Consider the usual unital commutative C^* -algebra $C[0, 1]$ and the automorphism $\alpha(f)(x) = f(1 - x)$. Then $C[0, 1]$ together with the indefinite involution $f^\#(x) = \overline{f(1 - x)}$ is a Krein C^* -algebra.

Example. Let (\mathcal{H}, J) be a Krein C^* -module. For each $T \in \mathcal{L}(\mathcal{H})$ there exists an operator $T^\# \in \mathcal{L}(\mathcal{H})$ such that $[T\xi, \eta] = [\xi, T^\#\eta]$. Evidently $T^\# = JT^*J$ and $\mathcal{L}(\mathcal{H})$ equipped with $\alpha(T) = JTJ$ is a Krein C^* -algebra.

Definition 1.3. Let \mathcal{A} be a C^* -algebra and let (\mathcal{K}, J) be a Krein C^* -module. A homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{K})$ is called a representation of \mathcal{A} on (\mathcal{K}, J) if $\pi(a^*) = J\pi(a)^*J = \pi(a)^\#$, or equivalently, $[\pi(a)\xi, \eta] = [\xi, \pi(a^*)\eta]$.

Focusing on the structure of indefinite version of Hilbert C^* -modules, we investigate Φ -maps associated to φ -maps, which are a certain type of α -completely positive maps. We then prove a KSGNS (Kasparov–Stinespring–Gel’fand–Naimark–Segal) type theorem for α -completely positive maps on Krein C^* -modules and show that the minimal KSGNS construction is unique up to unitary equivalence. We also establish a covariant version of the KSGNS type theorem for a covariant α -completely positive map and study the structure of minimal covariant KSGNS constructions. Our result provide some variants and some generalizations of results of [15] in the context of maps on Krein C^* -modules.

2. KSGNS TYPE CONSTRUCTION FOR α -CP MAPS

In this section, we assume that (\mathcal{A}, α) is a unital C^* -algebra with the unit 1. We start our work with the following modified definition of [12, Definition 2.4] playing an essential role in the paper. It provides a generalization of α -positivity introduced in Example 1.1.

Definition 2.1. An α -completely positive map (briefly, α -CP) of \mathcal{A} on a Krein C^* -module (\mathcal{H}, J) is a $*$ -map $\varphi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ such that

- (i) $\varphi(a^\#) = \varphi(a)^\# = \varphi(a^*)$, or equivalently, $\varphi(\alpha(a)) = J\varphi(a)J = \varphi(a)$ for all $a \in \mathcal{A}$;
- (ii) the $n \times n$ matrix $[\varphi(a_i^\# a_j)]$ is positive for all $n \geq 1$ and each $a_1, \dots, a_n \in \mathcal{A}$, or equivalently, $\sum_{i=1}^n \sum_{j=1}^n \langle \xi_i, \varphi(\alpha(a_i)^* a_j) \xi_j \rangle \geq 0$ for all $n \geq 1$, $a_1, \dots, a_n \in \mathcal{A}$ and $\xi_1, \dots, \xi_n \in \mathcal{H}$;
- (iii) for any $a \in \mathcal{A}$, there is $M(a) > 0$ such that

$$\left[\varphi \left((aa_i)^\# aa_j \right) \right]_{i,j=1}^n \leq M(a) \left[\varphi \left(a_i^\# a_j \right) \right]_{i,j=1}^n$$

for all $n \geq 1$ and $a_1, \dots, a_n \in \mathcal{A}$.

To be sure that our maps are continuous, we may assume that the constant $M(a)$ is of the form $K(a) \|a\|$ with $K(a) > 0$

Let (\mathcal{H}_1, J_1) and (\mathcal{H}_2, J_2) be Krein C^* -modules. For $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, let us put

$$T^\# = J_1 T^* J_2.$$

Definition 2.2. Let \mathcal{X} be a Hilbert \mathcal{A} -module and let $(\mathcal{H}_1, J_1), (\mathcal{H}_2, J_2)$ Krein \mathcal{B} -modules. For an α -CP map $\varphi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_1)$, a map $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is called a (α -completely positive) φ -map if for any $x, y \in \mathcal{X}$,

$$\Phi(x)^\# \Phi(y) = \varphi(\langle x, y \rangle).$$

Definition 2.3. A representation of a Hilbert \mathcal{A} -module \mathcal{X} on Krein \mathcal{B} -modules (\mathcal{H}_1, J_1) and (\mathcal{H}_2, J_2) is a map $\pi_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ with the property that there is a representation $\pi_{\mathcal{A}}$ of \mathcal{A} on (\mathcal{H}_1, J_1) such that

$$\pi_{\mathcal{X}}(x)^\# \pi_{\mathcal{X}}(y) = \pi_{\mathcal{A}}(\langle x, y \rangle).$$

Then we say that $\pi_{\mathcal{X}}$ is a $\pi_{\mathcal{A}}$ -representation.

Remark 2.4. Let $\pi_{\mathcal{X}}$ be a $\pi_{\mathcal{A}}$ -representation of a Hilbert \mathcal{A} -module \mathcal{X} on Krein \mathcal{B} -modules (\mathcal{H}_1, J_1) and (\mathcal{H}_2, J_2) .

- (1) If \mathcal{X} is full, then $\pi_{\mathcal{A}}$ is unique.
- (2) If $[\pi_{\mathcal{X}}(\mathcal{X})\mathcal{H}_1] = \mathcal{H}_2$, then for any $x \in \mathcal{X}$ and $a \in \mathcal{A}$,

$$\pi_{\mathcal{X}}(xa) = \pi_{\mathcal{X}}(x) \pi_{\mathcal{A}}(a). \quad (2.1)$$

Indeed, for each $x, y \in \mathcal{X}$ and $a \in \mathcal{A}$, we obtain that

$$\begin{aligned} \pi_{\mathcal{X}}(xa)^\# \pi_{\mathcal{X}}(y) &= \pi_{\mathcal{A}}(\langle xa, y \rangle) = \pi_{\mathcal{A}}(a^*) \pi_{\mathcal{A}}(\langle x, y \rangle) \\ &= \pi_{\mathcal{A}}(a)^\# \pi_{\mathcal{X}}(x)^\# \pi_{\mathcal{X}}(y) \\ &= (\pi_{\mathcal{X}}(x) \pi_{\mathcal{A}}(a))^\# \pi_{\mathcal{X}}(y), \end{aligned}$$

whence we deduce that $\pi_{\mathcal{X}}(xa)^\# = (\pi_{\mathcal{X}}(x) \pi_{\mathcal{A}}(a))^\#$, which implies (2.1).

The next proposition gives a typical example of an α -completely positive map.

Proposition 2.5. Let (\mathcal{A}, α) be a unital Krein C^* -algebra, $\pi_{\mathcal{A}}$ be a representation of \mathcal{A} on Krein C^* -modules $(\mathcal{K}_1, J_1 = \text{id}_{\mathcal{K}_1})$, let $\pi_{\mathcal{X}}$ be a $\pi_{\mathcal{A}}$ -representation of a Hilbert \mathcal{A} -module \mathcal{X} on $(\mathcal{K}_1, J_1 = \text{id}_{\mathcal{K}_1})$ and $(\mathcal{K}_2, J_2 = \text{id}_{\mathcal{K}_2})$, let (\mathcal{H}_1, J_3) and $(\mathcal{H}_2, J_4 = \text{id}_{\mathcal{H}_2})$ be Krein C^* -modules, and $V : \mathcal{H}_1 \rightarrow \mathcal{K}_1$ and $W : \mathcal{H}_2 \rightarrow \mathcal{K}_2$ be two operators such that $V^\# = V^*$, $\pi_{\mathcal{A}}(\alpha(a))V = J_1 \pi_{\mathcal{A}}(a) V J_3$ for all $a \in \mathcal{A}$, and finally let W be a coisometry with $W^\# = W^*$. Then the map $\varphi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_1)$ given by

$$\varphi(a) = V^\# \pi_{\mathcal{A}}(a) V \quad (a \in \mathcal{A})$$

is an α -completely positive map and the map $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ given by

$$\Phi(x) = W^\# \pi_{\mathcal{X}}(x) V \quad (x \in \mathcal{X})$$

is an α -completely positive φ -map.

Proof. Indeed, we have

$$\varphi(a^*) = V^\# \pi_{\mathcal{A}}(a^*) V = V^\# \pi_{\mathcal{A}}(a)^* V = (V^\# \pi_{\mathcal{A}}(a) V)^* = \varphi(a)^*$$

for all $a \in \mathcal{A}$ and

$$\varphi(\alpha(a)) = V^\# \pi_{\mathcal{A}}(\alpha(a)) V = V^\# J_1 \pi_{\mathcal{A}}(a) V J_3 = J_3 V^\# \pi_{\mathcal{A}}(a) V J_3 = J_3 \varphi(a) J_3$$

for all $a \in \mathcal{A}$. On the other hand,

$$\varphi(\alpha(a)) = V^\# \pi_{\mathcal{A}}(\alpha(a)) V = V^\# \pi_{\mathcal{A}}(a) J_1 V J_3 = V^\# \pi_{\mathcal{A}}(a) V = \varphi(a)$$

for all $a \in \mathcal{A}$. Therefore, $\varphi(\alpha(a)) = J_3\varphi(a)J_3 = \varphi(a)$ for all $a \in \mathcal{A}$. Now, let $a_1, \dots, a_n \in \mathcal{A}$ and $\xi_1, \dots, \xi_n \in \mathcal{H}_1$. Then we have

$$\begin{aligned}
\sum_{i=1}^n \sum_{j=1}^n \langle \xi_i, \varphi(\alpha(a_i)^* a_j) \xi_j \rangle &= \sum_{i=1}^n \sum_{j=1}^n \langle \xi_i, V^\# \pi_{\mathcal{A}}(\alpha(a_i)^* a_j) V \xi_j \rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n \langle \pi_{\mathcal{A}}(\alpha(a_i)^*)^* V \xi_i, \pi_{\mathcal{A}}(a_j) V \xi_j \rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n \langle J_1 \pi_{\mathcal{A}}(\alpha(a_i)) J_1 V \xi_i, \pi_{\mathcal{A}}(a_j) V \xi_j \rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n \langle \pi_{\mathcal{A}}(a_i) V J_3 \xi_i, \pi_{\mathcal{A}}(a_j) V \xi_j \rangle \\
&= \left\langle \sum_{i=1}^n \pi_{\mathcal{A}}(a_i) V \xi_i, \sum_{j=1}^n \pi_{\mathcal{A}}(a_j) V \xi_j \right\rangle \geq 0.
\end{aligned}$$

Let all $n \geq 1$ and $a, a_1, \dots, a_n \in \mathcal{A}$. Then

$$\begin{aligned}
&\left\langle \left[\varphi((aa_i)^\# aa_j) \right]_{i,j=1}^n (\xi_k)_{k=1}^n, (\xi_k)_{k=1}^n \right\rangle \\
&= \left\langle \left(\sum_{j=1}^n V^\# \pi_{\mathcal{A}}(\alpha(a_i^* a^*) aa_j) V \xi_j \right)_{i=1}^n, (\xi_k)_{k=1}^n \right\rangle \\
&= \sum_{i=1}^n \left\langle \sum_{j=1}^n V^\# \pi_{\mathcal{A}}(\alpha(a_i^* a^*) aa_j) V \xi_j, \xi_i \right\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n \langle \pi_{\mathcal{A}}(\alpha(a^*) aa_j) V \xi_j, \pi_{\mathcal{A}}(\alpha(a_i^*))^* V \xi_i \rangle \\
&= \left\langle \pi_{\mathcal{A}}(\alpha(a^*) a) \sum_{j=1}^n \pi_{\mathcal{A}}(a_j) V \xi_j, \sum_{i=1}^n \pi_{\mathcal{A}}(\alpha(a_i)) V \xi_i \right\rangle \\
&= \left\langle \pi_{\mathcal{A}}(\alpha(a^*) a) \sum_{j=1}^n \pi_{\mathcal{A}}(a_j) V \xi_j, \sum_{i=1}^n J_1 \pi_{\mathcal{A}}(a_i) V J_3 \xi_i \right\rangle \\
&= \left\langle \pi_{\mathcal{A}}(\alpha(a^*) a) \sum_{j=1}^n \pi_{\mathcal{A}}(a_j) V \xi_j, \sum_{i=1}^n \pi_{\mathcal{A}}(a_i) V \xi_i \right\rangle \\
&\leq \|\pi_{\mathcal{A}}(\alpha(a^*) a)\| \left\langle \sum_{j=1}^n \pi_{\mathcal{A}}(a_j) V \xi_j, \sum_{i=1}^n \pi_{\mathcal{A}}(a_i) V \xi_i \right\rangle \\
&\leq \|\pi_{\mathcal{A}}(\alpha(a^*) a)\| \left\langle \left[\varphi((a_i)^\# a_j) \right]_{i,j=1}^n (\xi_k)_{k=1}^n, (\xi_k)_{k=1}^n \right\rangle
\end{aligned}$$

for all $\xi_1, \xi_2, \dots, \xi_n \in \mathcal{H}_1$. Thus we showed that φ is an α -completely positive map.

To show that Φ is an α -completely positive φ -map, let $x, y \in \mathcal{A}$. We have

$$\begin{aligned}\Phi(x)^\# \Phi(y) &= J_3 V^\# \pi_{\mathcal{X}}(x)^* W J_4 W^\# \pi_{\mathcal{X}}(y) V = V^\# J_1 \pi_{\mathcal{X}}(\langle x, y \rangle) V \\ &= V^\# \pi_{\mathcal{X}}(\langle x, y \rangle) V = \varphi(\langle x, y \rangle).\end{aligned}$$

□

From now on we assume that (\mathcal{H}_1, J_1) and $(\mathcal{H}_2, J_2 = \text{id}_{\mathcal{H}_2})$ are Krein \mathcal{B} -modules, and $\varphi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_1)$ is an α -CP map. The next Theorem can be regarded as a generalization in the context of maps on Krein C^* -modules of [12, Theorem 4.4].

Theorem 2.6. *Let \mathcal{X} be a Hilbert \mathcal{A} -module and $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ a φ -map. Then there are a Krein \mathcal{B} -module (\mathcal{K}_1, J_3) and a Hilbert \mathcal{B} -module \mathcal{K}_2 , a representation π_φ of \mathcal{A} on (\mathcal{K}_1, J_1) , a π_φ -representation $\pi_{\mathcal{X}}$ of \mathcal{X} on (\mathcal{K}_1, J_3) and $(\mathcal{K}_2, J_4 = \text{id}_{\mathcal{K}_2})$, two operators $V_\Phi : \mathcal{H}_1 \rightarrow \mathcal{K}_1$ and $W_\Phi : \mathcal{H}_2 \rightarrow \mathcal{K}_2$ such that*

- (1) $V_\Phi^\# = V_\Phi^*$, $\pi_\varphi(\alpha(a))V_\Phi = J_3 \pi_\varphi(a) V_\Phi J_1$ for all $a \in \mathcal{A}$, and W_Φ is a coisometry with $W_\Phi^\# = W_\Phi^*$;
- (2) $\varphi(a) = V_\Phi^\# \pi_\varphi(a) V_\Phi$ for all $a \in \mathcal{A}$;
- (3) $\Phi(x) = W_\Phi^\# \pi_{\mathcal{X}}(x) V_\Phi$ for all $x \in \mathcal{X}$.

Proof. It is straightforward to observe via condition (ii) in Definition 2.1 that the quotient $\mathcal{A} \otimes_{\text{alg}} \mathcal{H}_1 / N_\varphi$ of the algebraic tensor product $\mathcal{A} \otimes_{\text{alg}} \mathcal{H}_1$, where

$$N_\varphi = \left\{ \sum_{i=1}^n a_i \otimes \xi_i : \sum_{i=1}^n \sum_{j=1}^n \langle \xi_i, \varphi(\alpha(a_i)^* a_j) \xi_j \rangle = 0 \right\},$$

$$(a \otimes \xi + N_\varphi)b = a \otimes (\xi b) + N_\varphi,$$

is a pre-Hilbert \mathcal{B} -module with the inner product given by

$$\left\langle \sum_{i=1}^n a_i \otimes \xi_i + N_\varphi, \sum_{j=1}^m b_j \otimes \eta_j + N_\varphi \right\rangle = \sum_{i=1}^n \sum_{j=1}^m \langle \xi_i, \varphi(\alpha(a_i)^* b_j) \eta_j \rangle.$$

Let \mathcal{K}_1 be the Hilbert \mathcal{B} -module obtained by the completion of $\mathcal{A} \otimes_{\text{alg}} \mathcal{H}_1 / N_\varphi$. Using 2.1 (i), it is easy to check that (\mathcal{K}_1, J_3) is a Krein \mathcal{B} -module, where the fundamental symmetry J_3 is defined by

$$J_3(a \otimes \xi + N_\varphi) = \alpha(a) \otimes (J_1 \xi) + N_\varphi.$$

Also it is easy to check that the map $V_\Phi : \mathcal{H}_1 \rightarrow \mathcal{K}_1$ defined by

$$V_\Phi \xi = 1 \otimes J_1 \xi + N_\varphi$$

is an operator, and $V_\Phi^*(a \otimes \xi + N_\varphi) = J_1 \varphi(a) \xi$. Moreover, for any $\xi \in \mathcal{H}_1$, $J_3 V_\Phi J_1(\xi) = J_3(1 \otimes \xi + N_\varphi) = 1 \otimes (J_1 \xi) + N_\varphi = V_\Phi(\xi)$, which implies that $J_3 V_\Phi J_1 = V_\Phi$, and so $V_\Phi^\# = V_\Phi^*$. Also the map $\pi_\varphi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{K}_1)$ given by

$$\pi_\varphi(a)(b \otimes \xi + N_\varphi) = ab \otimes \xi + N_\varphi \quad (a, b \in \mathcal{A}, \xi \in \mathcal{H}_1) \quad (2.2)$$

is a representation of \mathcal{A} on (\mathcal{K}_1, J_3) , in this case, we have $\pi_\varphi(a)^* = \pi_\varphi(\alpha(a)^*)$ (for details see [12, Theorem 4.4]). Since $V_\Phi^\# = V_\Phi^*$, we have

$$V_\Phi^\# \pi_\varphi(a) V_\Phi \xi = V_\Phi^*(a \otimes J_1(\xi) + N_\varphi) = J_1 \varphi(a) J_1 \xi = \varphi(a) \xi,$$

for all $\xi \in \mathcal{H}_1$, and so

$$\varphi(a) = V_{\Phi}^{\#} \pi_{\varphi}(a) V_{\Phi} \quad (a \in \mathcal{A}).$$

Moreover, $\mathcal{K}_1 = [\pi_{\varphi}(\mathcal{A}) V_{\Phi} \mathcal{H}_1]$. For any $a \in \mathcal{A}$ and $\xi \in \mathcal{H}_1$, we get

$$\pi_{\varphi}(\alpha(a)) V_{\Phi} \xi = \alpha(a) \otimes J_1(\xi) + N_{\varphi}$$

and

$$J_3 \pi_{\varphi}(a) V_{\Phi} J_1 \xi = J_3(a \otimes \xi + N_{\varphi}) = \alpha(a) \otimes J_1(\xi) + N_{\varphi}$$

whence

$$\pi_{\varphi}(\alpha(a)) V_{\Phi} = J_3 \pi_{\varphi}(a) V_{\Phi} J_1 \quad (a \in \mathcal{A}).$$

Let $\mathcal{K}_2 = [\Phi(\mathcal{X}) \mathcal{H}_1] \subseteq \mathcal{H}_2$ be the closed linear subspace. Then \mathcal{K}_2 is a Hilbert \mathcal{B} -module. For the inclusion $\mathbf{J}_{\mathcal{K}_2}$ of \mathcal{K}_2 into \mathcal{H}_2 , put $W_{\Phi}^* = \mathbf{J}_{\mathcal{K}_2}$ and then W_{Φ} is a coisometry and $W_{\Phi}^{\#} = W_{\Phi}^*$. On the other hand, for any $a, b \in \mathcal{A}$ and $x \in \mathcal{X}$, since $J_1 V_{\Phi}^{\#} = J_1 V_{\Phi}^* = V_{\Phi}^* J_3 = V_{\Phi}^{\#} J_3$, we obtain

$$\begin{aligned} \Phi(xa)^{\#} \Phi(xb) &= \varphi(\langle xa, xb \rangle) = \varphi(a^* \langle x, x \rangle b) \\ &= V_{\Phi}^{\#} \pi_{\varphi}(a^*) \pi_{\varphi}(\langle x, x \rangle b) V_{\Phi} \\ &= V_{\Phi}^{\#} J_3 \pi_{\varphi}(a)^* J_3 \pi_{\varphi}(\langle x, x \rangle b) V_{\Phi} \\ &= J_1 V_{\Phi}^{\#} \pi_{\varphi}(a)^* J_3 \pi_{\varphi}(\langle x, x \rangle b) V_{\Phi}. \end{aligned}$$

Therefore, for any $a_1, \dots, a_n \in \mathcal{A}$ and $\xi_1, \dots, \xi_n \in \mathcal{H}_1$, we have

$$\begin{aligned} \left\| \sum_{i=1}^n \Phi(xa_i) \xi_i \right\|^2 &= \left\| \sum_{i=1}^n \sum_{j=1}^n \left\langle \xi_i, J_1 \Phi(xa_i)^{\#} \Phi(xa_j) \xi_j \right\rangle \right\|^2 \\ &= \left\| \sum_{i=1}^n \sum_{j=1}^n \langle \pi_{\varphi}(a_i) V_{\Phi} \xi_i, J_3 \pi_{\varphi}(\langle x, x \rangle) \pi_{\varphi}(a_j) V_{\Phi} \xi_j \rangle \right\|^2 \\ &= \left\| \left\langle \sum_{i=1}^n \pi_{\varphi}(a_i) V_{\Phi} \xi_i, J_3 \pi_{\varphi}(\langle x, x \rangle) \sum_{j=1}^n \pi_{\varphi}(a_j) V_{\Phi} \xi_j \right\rangle \right\|^2 \\ &\leq \left\| \sum_{i=1}^n \pi_{\varphi}(a_i) V_{\Phi} \xi_i \right\|^2 \|J_3 \pi_{\varphi}(\langle x, x \rangle)\|, \end{aligned}$$

which implies that for each $x \in \mathcal{X}$, there exists a map $\pi_{\mathcal{X}}(x) : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ such that

$$\pi_{\mathcal{X}}(x) \left(\sum_{i=1}^n \pi_{\varphi}(a_i) V_{\Phi} \xi_i \right) = \sum_{i=1}^n \Phi(xa_i) \xi_i. \quad (2.3)$$

Then we obtain that

$$\begin{aligned}
& \left\langle \pi_{\mathcal{X}}(x) \left(\sum_{i=1}^n \pi_{\varphi}(a_i) V_{\Phi} \xi_i \right), \sum_{j=1}^m \Phi(y_j) \eta_j \right\rangle \\
&= \left\langle \sum_{i=1}^n \Phi(x a_i) \xi_i, \sum_{j=1}^m \Phi(y_j) \eta_j \right\rangle \\
&= \sum_{j=1}^m \sum_{i=1}^n \left\langle \xi_i, J_1 \Phi(x a_i)^{\#} \Phi(y_j) \eta_j \right\rangle \\
&= \sum_{j=1}^m \sum_{i=1}^n \left\langle \xi_i, V_{\Phi}^{\#} \pi_{\varphi}(a_i)^* J_3 \pi_{\varphi}(\langle x, y_j \rangle) V_{\Phi} \eta_j \right\rangle \\
&= \left\langle \sum_{i=1}^n \pi_{\varphi}(a_i) V_{\Phi} \xi_i, \sum_{j=1}^m J_3 \pi_{\varphi}(\langle x, y_j \rangle) V_{\Phi} \eta_j \right\rangle,
\end{aligned}$$

which implies that

$$\pi_{\mathcal{X}}(x)^* \left(\sum_{j=1}^m \Phi(y_j) \eta_j \right) = \sum_{j=1}^m J_3 \pi_{\varphi}(\langle x, y_j \rangle) V_{\Phi} \eta_j.$$

Therefore, $\pi_{\mathcal{X}}(x) \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. In this way we have obtained a map $\pi_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$, and then for any $a_1, \dots, a_n \in \mathcal{A}$, $\xi_1, \dots, \xi_n \in \mathcal{H}_1$ and $x, y \in \mathcal{X}$, we get

$$\begin{aligned}
\pi_{\mathcal{X}}(x)^{\#} \pi_{\mathcal{X}}(y) \left(\sum_{i=1}^n \pi_{\varphi}(a_i) V_{\Phi} \xi_i \right) &= J_3 \pi_{\mathcal{X}}(x)^* \left(\sum_{i=1}^n \Phi(y a_i) \xi_i \right) \\
&= J_3 \left(\sum_{i=1}^n J_3 \pi_{\varphi}(\langle x, y \rangle a_i) V_{\Phi} \xi_i \right) \\
&= \pi_{\varphi}(\langle x, y \rangle) \left(\sum_{i=1}^n \pi_{\varphi}(a_i) V_{\Phi} \xi_i \right),
\end{aligned}$$

which implies that for any $x, y \in \mathcal{X}$,

$$\pi_{\mathcal{X}}(x)^{\#} \pi_{\mathcal{X}}(y) = \pi_{\varphi}(\langle x, y \rangle). \quad (2.4)$$

Therefore $\pi_{\mathcal{X}}$ is a π_{φ} -representation of \mathcal{X} on the Krein spaces (\mathcal{K}_1, J_3) and (\mathcal{K}_2, J_4) . Moreover,

$$W_{\Phi}^{\#} \pi_{\mathcal{X}}(x) V_{\Phi} \xi = W_{\Phi}^* \Phi(x) \xi = \Phi(x) \xi$$

for all $\xi \in \mathcal{H}_1$. □

Remark 2.7. In Theorem 2.6, if φ is unital, i.e., $\varphi(1) = 1$, then since $V_{\Phi}^*(a \otimes \xi + N_{\varphi}) = J_1 \varphi(a) \xi$ for any $a \in \mathcal{A}$ and $\xi \in \mathcal{H}_1$, $V_{\Phi}^* V_{\Phi} \xi = V_{\Phi}^*(1 \otimes J_1 \xi + N_{\varphi}) = J_1 \varphi(1) J_1 \xi = \xi$, which implies that V_{Φ} is isometry.

A six-tuple $(\pi_{\mathcal{X}}, \pi_{\varphi}, V_{\Phi}, W_{\Phi}, (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_4))$, which verifying the relations (1)-(3) in Theorem 2.6 is called the KSGNS construction associated to the φ -map Φ . If $\mathcal{K}_2 = [\pi_{\mathcal{X}}(\mathcal{X}) V_{\varphi} \mathcal{H}_1]$ and $\mathcal{K}_1 = [\pi_{\varphi}(\mathcal{A}) V_{\Phi} \mathcal{H}_1]$, we say that six-tuple $(\pi_{\mathcal{X}}, \pi_{\varphi}, V_{\Phi}, W_{\Phi}, (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_4))$ is *minimal*.

Remark 2.8. The KSGNS construction associated to the φ -map Φ constructed in Theorem 2.6 is minimal. Using (2.3) we have

$$[\pi_{\mathcal{X}}(\mathcal{X}) V_{\Phi} \mathcal{H}_1] = [\pi_{\mathcal{X}}(\mathcal{X}) \pi_{\varphi}(1) V_{\Phi} \mathcal{H}_1] = [\Phi(\mathcal{X} \alpha(1)) \mathcal{H}_1] = [\Phi(\mathcal{X}) \mathcal{H}_1] = \mathcal{K}_2$$

and by applying (2.2) we get

$$[\pi_{\varphi}(\mathcal{A}) V_{\Phi} \mathcal{H}_1] = \mathcal{K}_1.$$

Remark 2.9. If $(\pi_{\mathcal{X}}, \pi_{\varphi}, V_{\Phi}, W_{\Phi}, (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_4))$ is a minimal KSGNS construction associated to the φ -map Φ , then $\mathcal{K}_1 = [\pi_{\varphi}(\mathcal{A})^* V_{\Phi} \mathcal{H}_1]$. Indeed, we obtain that

$$\begin{aligned} [\pi_{\varphi}(\mathcal{A})^* V_{\Phi} \mathcal{H}_1] &= [J_3 \pi_{\varphi}(\mathcal{A}^*) J_3 V_{\Phi} J_1 \mathcal{H}_1] = [J_3 \pi_{\varphi}(\mathcal{A}) V_{\Phi} \mathcal{H}_1] \\ &= J_3 [\pi_{\varphi}(\mathcal{A}) V_{\Phi} \mathcal{H}_1] = J_3 \mathcal{K}_1 = \mathcal{K}_1. \end{aligned}$$

The following result may be considered as a generalization in the context of maps on Krein C^* -modules of [12, Theorem 4.6].

Proposition 2.10. *Let \mathcal{X} be a Hilbert \mathcal{A} -module and $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a φ -map. If the constructions $(\pi_{\mathcal{X}}, \pi_{\varphi}, V_{\Phi}, W_{\Phi}, (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_4))$ and $(\pi'_{\mathcal{X}}, \pi'_{\varphi}, V'_{\Phi}, W'_{\Phi}, (\mathcal{K}'_1, J'_3), (\mathcal{K}'_2, J'_4))$ are two minimal KSGNS construction for Φ , then there are two unitary operators $U_1 : \mathcal{K}_1 \rightarrow \mathcal{K}'_1$ and $U_2 : \mathcal{K}_2 \rightarrow \mathcal{K}'_2$ such that*

- (1) $U_1 V_{\Phi} = V'_{\Phi}$ and $U_1 \pi_{\varphi}(a) U_1^{\#} = \pi'_{\varphi}(a)$ for all $a \in \mathcal{A}$;
- (2) $U_2 W_{\Phi} = W'_{\Phi}$ and $U_2 \pi_{\mathcal{X}}(x) U_2^{\#} = \pi'_{\mathcal{X}}(x)$ for all $x \in \mathcal{X}$.

Proof. From Theorem (1) 2.6, we have

$$\begin{aligned} &\left\langle \sum_{i=1}^n \pi'_{\varphi}(a_i) V'_{\Phi} \xi_i, \sum_{j=1}^m \pi'_{\varphi}(b_j) V'_{\Phi} \eta_j \right\rangle \\ &= \left\langle \sum_{i=1}^n V'_{\Phi} \xi_i, \sum_{j=1}^m V'^{*}_{\Phi} \pi'_{\varphi}(a_i)^* \pi'_{\varphi}(b_j) V'_{\Phi} \eta_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \left\langle \xi_i, (V'_{\Phi})^{\#} J_3 \pi'_{\varphi}(a_i^*) J_3 \pi'_{\varphi}(b_j) V'_{\Phi} \eta_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \left\langle \xi_i, (V'_{\Phi})^{\#} J_3 \pi'_{\varphi}(a_i^*) \pi'_{\varphi}(\alpha(b_j)) V'_{\Phi} J_1 \eta_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \left\langle \xi_i, (V'_{\Phi})^{\#} \pi'_{\varphi}(\alpha(a_i^*) b_j) V'_{\Phi} \eta_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \left\langle \xi_i, \varphi(a_i^{\#} b_j) \eta_j \right\rangle \\ &= \left\langle \sum_{i=1}^n \pi_{\varphi}(a_i) V_{\Phi} \xi_i, \sum_{j=1}^m \pi_{\varphi}(b_j) V_{\Phi} \eta_j \right\rangle, \end{aligned}$$

which implies that there is a unitary operator $U_1 : \mathcal{K}_1 \rightarrow \mathcal{K}'_1$ such that

$$U_1 \left(\sum_{i=1}^n \pi_\varphi(a_i) V_\Phi \xi_i \right) = \sum_{i=1}^n \pi'_\varphi(a_i) V'_\Phi \xi_i.$$

Then it is easy to check that $U_1 V_\Phi = V'_\Phi$ and $U_1 \pi_\varphi(a) = \pi'_\varphi(a) U_1$ for all $a \in \mathcal{A}$. Also, for each $a_1, \dots, a_n, b_1, \dots, b_m \in \mathcal{A}$, $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m \in \mathcal{H}_1$ and each $a \in \mathcal{A}$ we obtain that

$$\begin{aligned} U_1 \pi_\varphi(a) U_1^\# \left(\sum_{i=1}^n \pi'_\varphi(a_i) V'_\Phi \xi_i \right) &= U_1 \pi_\varphi(a) J_3 U_1^* J'_3 \left(\sum_{i=1}^n \pi'_\varphi(a_i) V'_\Phi \xi_i \right) \\ &= U_1 \pi_\varphi(a) J_3 U_1^* \left(\sum_{i=1}^n \pi'_\varphi(\alpha(a_i)) V'_\Phi J_1 \xi_i \right) \\ &= U_1 \pi_\varphi(a) J_3 \left(\sum_{i=1}^n \pi_\varphi(\alpha(a_i)) V'_\Phi J_1 \xi_i \right) \\ &= U_1 \pi_\varphi(a) \left(\sum_{i=1}^n \pi_\varphi(a_i) V_\Phi \xi_i \right) \\ &= \sum_{i=1}^n \pi'_\varphi(a a_i) V'_\Phi \xi_i \\ &= \pi'_\varphi(a) \left(\sum_{i=1}^n \pi'_\varphi(a_i) V'_\Phi \xi_i \right), \end{aligned}$$

and so $U_1 \pi_\varphi(a) U_1^\# = \pi'_\varphi(a)$ for all $a \in \mathcal{A}$. Also, from (3) of in Theorem 2.6 we observe that

$$\begin{aligned} &\left\langle \sum_{i=1}^n \pi'_\mathcal{X}(x_i) V'_\Phi \xi_i, \sum_{j=1}^m \pi'_\mathcal{X}(y_j) V'_\Phi \eta_j \right\rangle \\ &= \left\langle \sum_{i=1}^n (W'_\Phi)^\# \pi'_\mathcal{X}(x_i) V'_\Phi \xi_i, \sum_{j=1}^m (W'_\Phi)^\# \pi'_\mathcal{X}(y_j) V'_\Phi \eta_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \langle \Phi(x_i) \xi_i, \Phi(y_j) \eta_j \rangle \\ &= \left\langle \sum_{i=1}^n W_\Phi^\# \pi_\mathcal{X}(x_i) V_\Phi \xi_i, \sum_{j=1}^m W_\Phi^\# \pi_\mathcal{X}(y_j) V_\Phi \eta_j \right\rangle \\ &= \left\langle \sum_{i=1}^n \pi_\mathcal{X}(x_i) V_\Phi \xi_i, \sum_{j=1}^m \pi_\mathcal{X}(y_j) V_\Phi \eta_j \right\rangle, \end{aligned}$$

which implies that there is a unitary operator $U_2 : \mathcal{K}_2 \rightarrow \mathcal{K}'_2$ such that

$$U_2 \left(\sum_{i=1}^n \pi_\mathcal{X}(x_i) V_\Phi \xi_i \right) = \sum_{i=1}^n \pi'_\mathcal{X}(x_i) V'_\Phi \xi_i.$$

Then by using (3) of Theorem 2.6 we obtain that

$$\begin{aligned}
W_{\Phi}^* U_2^* \left(\sum_{i=1}^n \pi'_{\mathcal{X}}(x_i) V'_{\Phi} \xi_i \right) &= \sum_{i=1}^n W_{\Phi}^* \pi_{\mathcal{X}}(x_i) V_{\Phi} \xi_i = \sum_{i=1}^n W_{\Phi}^{\#} \pi_{\mathcal{X}}(x_i) V_{\Phi} \xi_i \\
&= \sum_{i=1}^n \Phi(x_i) \xi_i = \sum_{i=1}^n (W'_{\Phi})^{\#} \pi'_{\mathcal{X}}(x_i) V'_{\Phi} \xi_i \\
&= (W'_{\Phi})^* \sum_{i=1}^n \pi'_{\mathcal{X}}(x_i) V'_{\Phi} \xi_i
\end{aligned}$$

and, since $\mathcal{K}'_2 = [\pi'_{\mathcal{X}}(\mathcal{X})' V'_{\Phi} \mathcal{H}_1]$, and so $U_2 W_{\Phi} = W'_{\Phi}$. On the other hand, by applying (1) of Theorem 2.6 and (2.1) for any $a_1, \dots, a_n \in \mathcal{A}$, $\xi_1, \dots, \xi_n \in \mathcal{H}_1$ and $x \in \mathcal{X}$, we obtain that

$$\begin{aligned}
U_2 \pi_{\mathcal{X}}(x) U_1^{\#} \left(\sum_{i=1}^n \pi'_{\varphi}(a_i) V'_{\Phi} \xi_i \right) &= U_2 \pi_{\mathcal{X}}(x) J_3 U_1^* J_3' \left(\sum_{i=1}^n \pi'_{\varphi}(a_i) V'_{\Phi} \xi_i \right) \\
&= U_2 \pi_{\mathcal{X}}(x) \left(\sum_{i=1}^n \pi_{\varphi}(a_i) V_{\Phi} \xi_i \right) \\
&= U_2 \left(\sum_{i=1}^n \pi_{\mathcal{X}}(x a_i) V_{\Phi} \xi_i \right) \\
&= \sum_{i=1}^n \pi'_{\mathcal{X}}(x a_i) V_{\Phi} \xi_i \\
&= \pi'_{\mathcal{X}}(x) \left(\sum_{i=1}^n \pi'_{\varphi}(a_i) V'_{\Phi} \xi_i \right)
\end{aligned}$$

and taking into account that $\mathcal{K}_1 = [\pi'_{\varphi}(\mathcal{A}) V'_{\Phi} \mathcal{H}_1]$, we deduce that $\pi'_{\mathcal{X}}(x) = U_2 \pi_{\mathcal{X}}(x) U_1^{\#}$ for all $x \in \mathcal{X}$. \square

3. COVARIANT α -CP MAPS

Let \mathcal{G} be a locally compact group and let \mathcal{X} be a full Hilbert C^* -module over a unital C^* -algebra \mathcal{A} . An action of \mathcal{G} on \mathcal{X} is a group morphism η from \mathcal{G} to $\text{Aut}(\mathcal{X})$, the group of all Hilbert C^* -module isomorphisms from \mathcal{X} onto \mathcal{X} , such that the map $t \mapsto \eta_t(x)$ from \mathcal{G} to \mathcal{X} is continuous for each $x \in \mathcal{X}$. The triple $(\mathcal{G}, \eta, \mathcal{X})$ is called a dynamical system on Hilbert C^* -modules (see, [20, 18]).

An action $t \mapsto \eta_t$ of \mathcal{G} on \mathcal{X} induces a unique action $t \mapsto \beta_t^{\eta}$ of \mathcal{G} on \mathcal{A} such that $\beta_t^{\eta}(\langle x, y \rangle) = \langle \eta_t(x), \eta_t(y) \rangle$ for all $x, y \in \mathcal{X}, t \in \mathcal{G}$; see [20, 18].

A pseudo-unitary representation of \mathcal{G} on a Krein space (\mathcal{H}, J) is a map $t \mapsto u_t$ from \mathcal{G} to $\mathcal{L}(\mathcal{H})$ such that $u_e = \text{id}_{\mathcal{H}}$, $u_{ts} = u_t u_s$ and $u_{t^{-1}} = u_t^{\#}$ for all $s, t \in \mathcal{G}$. It follows from $u_{t^{-1}} = u_t^{\#}$ that $u_{t^{-1}} = J u_t^* J$. So

$$J u_{t^{-1}} = u_t^* J.$$

When we deal with usual Hilbert spaces $u_t^{\#}$ is replaced by u_t^* , and the pseudo-unitary representation is replaced by unitary representation.

Let $t \mapsto u_t$ and $t \mapsto u'_t$ be two pseudo-unitary $*$ -representations of \mathcal{G} on Krein spaces (\mathcal{H}_1, J_1) and (\mathcal{H}_2, J_2) .

Definition 3.1. A φ -map $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be (u', u) -covariant with respect to η if

$$\Phi(\eta_t(x)) = u'_t \Phi(x) u_t^\#$$

for all $t \in \mathcal{G}$, $x \in \mathcal{X}$, and

$$\beta_t^\eta \circ \alpha = \alpha \circ \beta_t^\eta$$

for all $t \in \mathcal{G}$.

Remark 3.2. Let $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a φ -map. If Φ is (u', u) -covariant with respect to η , then φ is u -covariant with respect to β^η , which means that

$$\varphi(\beta_t^\eta(a)) = u_t \varphi(a) u_t^* \quad (a \in \mathcal{A}, t \in \mathcal{G}).$$

In fact, for any $x, y \in \mathcal{X}$ and $t \in \mathcal{G}$, we obtain that

$$\varphi(\beta_t^\eta(\langle x, y \rangle)) = \varphi(\langle \eta_t(x), \eta_t(y) \rangle) = \Phi(\eta_t(x))^\# \Phi(\eta_t(y)) \quad (3.1)$$

$$= u_t \Phi(x)^\# (u'_t)^\# u'_t \Phi(y) u_t^\# \quad (3.2)$$

$$= u_t \varphi(\langle x, y \rangle) u_t^\#. \quad (3.3)$$

Let $t \mapsto u_t$ and $t \mapsto u'_t$ be two unitary representations of \mathcal{G} on Krein spaces (\mathcal{H}_1, J_1) and (\mathcal{H}_2, J_2) .

A representation $\pi_{\mathcal{X}}$ of \mathcal{X} on Krein spaces (\mathcal{H}_1, J_1) and (\mathcal{H}_2, J_2) is (u', u) -covariant with respect to η if

$$\pi_{\mathcal{X}}(\eta_t(x)) = u'_t \pi_{\mathcal{X}}(x) u_t^\#$$

for all $t \in \mathcal{G}$, $x \in \mathcal{X}$.

If $\pi_{\mathcal{X}}$ is (u', u) -covariant with respect to η , $u_t^\# = u_t^*$ for all $t \in \mathcal{G}$ and $\pi_{\mathcal{A}}$ is continuous, then $\pi_{\mathcal{A}}$ is u -covariant with respect to β^η in the sense that $\pi_{\mathcal{A}}(\beta_t^\eta(a)) = u_t \pi_{\mathcal{A}}(a) u_t^\#$ for all $a \in \mathcal{A}$, $t \in \mathcal{G}$. Indeed, by similar arguments as used in (3.3), we have

$$\begin{aligned} \pi_{\mathcal{A}}(\beta_t^\eta(\langle x, y \rangle)) &= u_t \pi_{\mathcal{X}}(x)^\# \pi_{\mathcal{X}}(y) u_t^\# \\ &= u_t \pi_{\mathcal{A}}(\langle x, y \rangle) u_t^\# \end{aligned}$$

for all $x, y \in \mathcal{X}$ and $t \in \mathcal{G}$, whence, since \mathcal{X} is full and $\pi_{\mathcal{A}}$ is continuous, we deduce that $\pi_{\mathcal{A}}$ is u -covariant with respect to β^η .

From now on, we assume that (\mathcal{H}_1, J_1) is a Krein space and u is simultaneously unitary and pseudo-unitary representation of \mathcal{G} on the Krein space (\mathcal{H}_1, J_1) , i.e.,

$$u_t^* u_t = u_t u_t^* = 1, \quad u_t^\# u_t = u_t u_t^\# = 1$$

for all $t \in \mathcal{G}$, which also implies that $u_t^* = u_t^\#$, equivalently, $J_1 u_t = u_t J_1$ for all $t \in \mathcal{G}$.

Lemma 3.3. Let \mathcal{X} be a Hilbert \mathcal{A} -module, $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a φ -map, which is (u', u) -covariant with respect to η and (\mathcal{K}_1, J_3) be the Krein space constructed in Theorem 2.6. Then there is a simultaneously unitary and pseudo-unitary representation v of \mathcal{G} on \mathcal{K}_1 such that

$$v_t(a \otimes \xi + N_\varphi) = \beta_t^\eta(a) \otimes u_t(\xi) + N_\varphi, \quad (3.4)$$

$$V_\Phi u_t = v_t V_\Phi \quad (3.5)$$

for any $a \in \mathcal{A}$, $\xi \in \mathcal{X}$ and $t \in \mathcal{G}$.

Proof. For any $t \in \mathcal{G}$, $a \in \mathcal{A}$ and $\xi \in \mathcal{H}_1$, we obtain that

$$\begin{aligned} \|\beta_t^\eta(a) \otimes u_t(\xi) + N_\varphi\|^2 &= \langle \xi, u_t^* \varphi(\beta_t^\eta(\alpha(a)^*) \beta_t^\eta(a)) u_t(\xi) \rangle \\ &= \left\langle \xi, u_t^* u_t \varphi(\alpha(a)^* b) u_t^\# u_t(\xi) \right\rangle \\ &= \langle a \otimes \xi + N_\varphi, a \otimes \xi + N_\varphi \rangle, \end{aligned}$$

which implies that there is an isometry operator $v_t : \mathcal{K}_1 \rightarrow \mathcal{K}_1$ such that (3.4) holds. By similar arguments, we can easily see that $v_t^*(a \otimes \xi) = \beta_{-t}^\eta(a) \otimes u_t \xi$ for all $a \in \mathcal{A}$ and $\xi \in \mathcal{H}_1$. Therefore, for any $t \in \mathcal{G}$, v_t is simultaneously unitary and pseudo-unitary. Moreover, we observe that

$$\begin{aligned} V_\Phi u_t(\xi) &= 1 \otimes J_1 u_t(\xi) + N_\varphi = \beta_t^\eta(1) \otimes u_t(J_1 \xi) + N_\varphi = v_t(1 \otimes J_1 \xi + N_\varphi) \\ &= v_t V_\Phi(\xi) \end{aligned}$$

for all $t \in \mathcal{G}$, $\xi \in \mathcal{H}_1$, and so we have (3.5). \square

The next result is a variant of [15, Theorem 3.2].

Theorem 3.4. *Let \mathcal{X} be a Hilbert \mathcal{A} -module, $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a φ -map, which is (u', u) -covariant with respect to η . Then there are two Krein spaces (\mathcal{K}_1, J_3) and (\mathcal{K}_2, J_4) , two pseudo-unitary representations v and v' of \mathcal{G} on the Krein spaces (\mathcal{K}_1, J_3) and (\mathcal{K}_2, J_4) , respectively, a representation π_φ of \mathcal{A} on (\mathcal{K}_1, J_3) , a π_φ -representation $\pi_\mathcal{X}$ of \mathcal{X} on (\mathcal{K}_1, J_3) and (\mathcal{K}_2, J_4) , which is (v', v) -covariant and two operators $V_\Phi : \mathcal{H}_1 \rightarrow \mathcal{K}_1$ and $W_\Phi : \mathcal{H}_2 \rightarrow \mathcal{K}_2$ such that*

- (1) $V_\Phi^\# = V_\Phi^*$, $\pi_\varphi(\alpha(a))V_\Phi = J_3 \pi_\varphi(a) V_\Phi J_1$ for all $a \in \mathcal{A}$, and W_Φ is a coisometry with $W_\Phi^\# = W_\Phi^*$;
- (2) $\varphi(a) = V_\Phi^\# \pi_\varphi(a) V_\Phi$ for all $a \in \mathcal{A}$;
- (3) $\Phi(x) = W_\Phi^\# \pi_\mathcal{X}(x) V_\Phi$ for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$;
- (4) $v_t^\# = v_t^*$ and $(v_t')^\# = (v_t')^*$ for all $t \in \mathcal{G}$;
- (5) $V_\Phi u_t = v_t V_\Phi$ and $W_\Phi u_t' = v_t' W_\Phi$ for all $t \in \mathcal{G}$.

Proof. Let $(\pi_\mathcal{X}, \pi_\varphi, V_\Phi, W_\Phi, (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_4))$ be the KSGNS construction associated to the φ -map Φ constructed in Theorem 2.6. Then by Lemma 3.3 there is a simultaneously unitary and pseudo-unitary representation v of \mathcal{G} on \mathcal{K}_1 satisfying (3.4) and (3.5). Moreover, π_φ is v -covariant with respect to β^η , since $v_t^\# = v_{-t} = v_t^*$ and

$$\begin{aligned} v_t \pi_\varphi(a) v_t^\#(b \otimes \xi + N_\varphi) &= v_t \pi_\varphi(a) (\beta_{t-1}^\eta(b) \otimes u_t^* \xi + N_\varphi) \\ &= v_t (a \beta_{t-1}^\eta(b) \otimes u_t^* \xi + N_\varphi) \\ &= \beta_t^\eta(a \beta_{t-1}^\eta(b)) \otimes u_t u_t^* \xi + N_\varphi \\ &= \beta_t^\eta(a) b \otimes \xi + N_\varphi \\ &= \pi_\varphi(\beta_t^\eta(a)) \end{aligned}$$

for all $a, b \in \mathcal{A}$, $\xi \in \mathcal{H}_1$ and $t \in \mathcal{G}$. Since Φ is (u', u) -covariant,

$$u'_t \left(\sum_{i=1}^n \Phi(x_i) \xi_i \right) = \sum_{i=1}^n \Phi(\eta_t(x_i)) u_t \xi_i$$

for all $t \in \mathcal{G}$, $x_i \in \mathcal{X}$ and $\xi_i \in \mathcal{H}_1$ ($i = 1, \dots, n$), hence $[\Phi(\mathcal{X})\mathcal{H}_1] = \mathcal{K}_2$ is invariant under u'_t as well as $(u'_t)^*$. Therefore, since W_Φ is the projection onto $[\Phi(\mathcal{X})\mathcal{H}_1]$, we have $u'_t W_\Phi = W_\Phi u'_t$ on \mathcal{K}_2 . Let $v'_t = u'_t|_{\mathcal{K}_2}$ for all $t \in \mathcal{G}$. Then $t \mapsto v'_t$ is a unitary representation of \mathcal{G} on \mathcal{K}_2 such that $W_\Phi u'_t = v'_t W_\Phi$ and since $J_4 = \text{id}_{\mathcal{K}_2}$, v' also can be considered as a pseudo-unitary representation of \mathcal{G} on (\mathcal{K}_2, J_4) . On the other hand, for any $t \in \mathcal{G}$ and $x \in \mathcal{X}$, we obtain that

$$\begin{aligned} v_t^\# \left(\sum_{i=1}^n \pi_\varphi(a_i) V_\Phi \xi_i \right) &= v_t^* \left(\sum_{i=1}^n a_i \otimes J_1 \xi_i \right) = \sum_{i=1}^n \beta_{t^{-1}}^\eta(a_i) \otimes u_t^* J_1 \xi_i \\ &= \sum_{i=1}^n \pi_\varphi(\beta_{t^{-1}}^\eta(a_i)) V_\Phi u_t^* \xi_i \end{aligned}$$

and, since $\eta_t(xa) = \eta_t(x) \beta_t^\eta(a)$, for any $a_1, \dots, a_n \in \mathcal{A}$ and $\xi_1, \dots, \xi_n \in \mathcal{H}_1$, we have

$$\begin{aligned} v'_t \pi_{\mathcal{X}}(x) v_t^\# \left(\sum_{i=1}^n \pi_\varphi(a_i) V_\Phi \xi_i \right) &= v'_t \pi_{\mathcal{X}}(x) \left(\sum_{i=1}^n \pi_\varphi(\beta_{t^{-1}}^\eta(a_i)) V_\Phi u_t^* \xi_i \right) \\ &= v'_t \left(\sum_{i=1}^n \Phi(x \beta_{t^{-1}}^\eta(a_i)) u_t^* \xi_i \right) \\ &= \sum_{i=1}^n \Phi(\eta_t(x \beta_{t^{-1}}^\eta(a_i))) u_t u_t^* \xi_i \\ &= \sum_{i=1}^n \Phi(\eta_t(x) a_i) \xi_i \\ &= \pi_{\mathcal{X}}(\eta_t(x)) \left(\sum_{i=1}^n \pi_\varphi(a_i) V_\Phi \xi_i \right), \end{aligned}$$

which implies that $\pi_{\mathcal{X}}$ is (v', v) -covariant, since $[\pi_\varphi(\mathcal{A}) V_\Phi \mathcal{H}_1] = \mathcal{K}_1$. \square

We say that $(\pi_\Phi, \pi_\varphi, V_\Phi, W_\Phi, v, v', (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_4))$ is a covariant KSGNS construction associated to the φ -map Φ being (u', u) -covariant with respect to η . If $\mathcal{K}_2 = [\pi_\Phi(\mathcal{X}) V_\Phi \mathcal{H}_1]$ and $\mathcal{K}_1 = [\pi_\varphi(\mathcal{A}) V_\Phi \mathcal{H}_1]$, we say that $(\pi_\Phi, \pi_\varphi, V_\Phi, W_\Phi, v, v', (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_4))$ is *minimal*.

The next proposition is a variant of [15, Theorem 3.5].

Proposition 3.5. *Suppose that \mathcal{X} is a Hilbert \mathcal{A} -module, $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is a φ -map and (u', u) -covariant with respect to η . If $(\pi_\Phi, \pi_\varphi, V_\Phi, W_\Phi, v, v', (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_4))$ and $(\pi'_\Phi, \pi'_\varphi, V'_\Phi, W'_\Phi, w, w', (\mathcal{K}'_1, J'_3), (\mathcal{K}'_2, J'_4))$ are two minimal covariant KSGNS constructions for Φ , then there exist two unitary operators $U_1 : \mathcal{K}_1 \rightarrow \mathcal{K}'_1$ and $U_2 : \mathcal{K}_2 \rightarrow \mathcal{K}'_2$ such that*

$$(1) \quad U_1 V_\Phi = V'_\Phi, \quad U_1 \pi_\varphi(a) U_1^\# = \pi'_\varphi(a) \text{ for all } a \in \mathcal{A} \text{ and } U_1 v_t U_1^\# = w_t \text{ for all } t \in \mathcal{G}$$

(2) $U_2 W_\Phi = W'_\Phi$, $U_2 \pi_{\mathcal{X}}(x) U_1^\# = \pi'_{\mathcal{X}}(x)$ for all $x \in \mathcal{X}$, and $U_2 v'_t U_2^\# = w'_t$ for all $t \in \mathcal{G}$.

Proof. It is easy to see that the representations $(\pi_{\mathcal{X}}, \pi_\varphi, V_\Phi, W_\Phi, (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_4))$ and $(\pi'_{\mathcal{X}}, \pi'_\varphi, V'_\Phi, W'_\Phi, (\mathcal{K}'_1, J'_3), (\mathcal{K}'_2, J'_4))$ are two minimal Stinespring representations for Φ . Let U_1 and U_2 the unitary operators defined in Proposition 2.10. Then for any $t \in \mathcal{G}$, $a_1, \dots, a_n \in \mathcal{A}$, $\xi_1, \dots, \xi_n \in \mathcal{H}_1$, we have

$$\begin{aligned} U_1 v_t U_1^\# \left(\sum_{i=1}^n \pi'_\varphi(a_i) V'_\Phi \xi_i \right) &= U_1 \left(\sum_{i=1}^n \pi_\varphi(\beta_t^\eta(a_i)) v_t V_\Phi \xi_i \right) \\ &= U_1 \left(\sum_{i=1}^n \pi_\varphi(\beta_t^\eta(a_i)) V_\Phi u_t \xi_i \right) \\ &= \sum_{i=1}^n w_t \pi'_\varphi(a_i) w_t^\# V'_\Phi u_t \xi_i \\ &= w_t \left(\sum_{i=1}^n \pi'_\varphi(a_i) V'_\Phi \xi_i \right), \end{aligned}$$

which implies that $U_1 v_t U_1^\# = w_t$ for all $t \in \mathcal{G}$. On the other hand, since $J_4 = \text{id}_{\mathcal{K}_2}$, from (2.3) we obtain that

$$\begin{aligned} v'_t U_2^\# (J'_2 \pi'_{\mathcal{X}}(x) V'_\Phi \xi) &= v'_t (\pi_{\mathcal{X}}(x) V_\Phi \xi) = v'_t (\Phi(x) \xi) \\ &= \Phi(\eta_t(x)) u_t \xi = \pi_{\mathcal{X}}(\eta_t(x)) V_\Phi u_t \xi \end{aligned}$$

and so from (2.4) for any $x_1, \dots, x_n, y_1, \dots, y_m \in \mathcal{X}$, $\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_m \in \mathcal{H}_1$ and $t \in \mathcal{G}$ we obtain that

$$\begin{aligned} &\left\langle U_2 v'_t U_2^\# \left(J'_2 \sum_{i=1}^n \pi'_{\mathcal{X}}(x_i) V'_\Phi \xi_i \right), \sum_{j=1}^m \pi'_{\mathcal{X}}(y_j) V'_\Phi \zeta_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \left\langle J_3 \pi_{\mathcal{X}}(y_j)^\# \pi_{\mathcal{X}}(\eta_t(x_i)) V_\Phi u_t \xi_i, V_\Phi \zeta_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \langle J_3 \pi_\varphi(\langle y_j, \eta_t(x_i) \rangle) V_\Phi u_t \xi_i, V_\Phi \zeta_j \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \left\langle V_\Phi^\# \pi_\varphi(\langle y_j, \eta_t(x_i) \rangle) V_\Phi u_t \xi_i, J_1 \zeta_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \langle \varphi(\langle y_j, \eta_t(x_i) \rangle) u_t \xi_i, J_1 \zeta_j \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \left\langle \pi'_{\mathcal{X}}(y_j)^\# \pi'_{\mathcal{X}}(\eta_t(x_i)) V'_\Phi u_t \xi_i, V'_\Phi J_1 \zeta_j \right\rangle \\ &= \left\langle w'_t \sum_{i=1}^n J'_2 \pi'_{\mathcal{X}}(x_i) V'_\Phi \xi_i, \sum_{j=1}^m \pi'_{\mathcal{X}}(y_j) V'_\Phi \zeta_j \right\rangle \end{aligned}$$

and then taking into account that $[J'_2 \pi'_\mathcal{X}(\mathcal{X}) V'_\Phi \mathcal{H}_1] = J'_2 [\pi'_\mathcal{X}(\mathcal{X}) V'_\Phi \mathcal{H}_1] = \mathcal{K}'_2$, we deduce that $U_2 v'_t U_2^\# = w'_t$ for all $t \in \mathcal{G}$. \square

Let \mathcal{G} be a locally compact group with a left Haar measure dt and the modular function $\Delta : \mathcal{G} \rightarrow (0, \infty)$. Let \mathcal{X} be a full Hilbert C^* -module over a unital C^* -algebra \mathcal{A} and η an action of \mathcal{G} on \mathcal{X} .

The linear space $C_c(\mathcal{G}, \mathcal{X})$ of all continuous functions from \mathcal{G} to \mathcal{X} with compact support has a structure of pre-Hilbert $\mathcal{G} \times_{\beta\eta} \mathcal{A}$ -module with the action of $\mathcal{G} \times_{\beta\eta} \mathcal{A}$ on $C_c(\mathcal{G}, \mathcal{X})$ given by

$$(\widehat{x}f)(s) = \int_{\mathcal{G}} \widehat{x}(t) \beta_t^\eta(f(t^{-1}s)) dt$$

for all $\widehat{x} \in C_c(\mathcal{G}, \mathcal{X})$, $f \in C_c(\mathcal{G}, \mathcal{A})$ and the inner product given by

$$\langle \widehat{x}, \widehat{y} \rangle(s) = \int_{\mathcal{G}} \beta_{t^{-1}}^\eta(\langle \widehat{x}(t), \widehat{y}(ts) \rangle) dt. \quad (3.6)$$

The crossed product of \mathcal{X} by η , denoted by $\mathcal{G} \times_\eta \mathcal{X}$, is the Hilbert $\mathcal{G} \times_{\beta\eta} \mathcal{A}$ -module obtained by the completion of the pre-Hilbert $\mathcal{G} \times_{\beta\eta} \mathcal{A}$ -module $C_c(\mathcal{G}, \mathcal{X})$ (see, for example, [20]).

Suppose that $\varphi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_1)$ is an α -CP map such that the constant $M(a)$ from Definition 2.1 (iii) is of the form $K(a) \|a\|$ with $K(a) > 0$.

If φ is u -covariant with respect to β^η , then there is a unique $\tilde{\alpha}$ -CP map $\tilde{\varphi} : \mathcal{G} \times_{\beta\eta} \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_1)$ such that

$$\tilde{\varphi}(f) = \int_{\mathcal{G}} \varphi(f(t)) u_t dt$$

for all $f \in C_c(\mathcal{G}, \mathcal{A})$, where $\tilde{\alpha}(f) = \alpha \circ f$ for all $f \in C_c(\mathcal{G}, \mathcal{A})$ (it is similar to [14, Theorem 4.3]). Moreover, if $(\pi_\varphi, V_\Phi, v_t, (\mathcal{K}_1, J_3))$ is the minimal KSGNS construction associated to φ , then $(\widehat{\pi_\varphi}, V_\Phi, (\mathcal{K}_1, J_3))$ is the minimal KSGNS construction associated to $\tilde{\varphi}$, where $\widehat{\pi_\varphi}(f) = \int_{\mathcal{G}} \pi_\varphi(f(t)) v_t dt$ for all $f \in C_c(\mathcal{G}, \mathcal{A})$. The next result may be compared with [15, Corollary 4.3].

Theorem 3.6. *Let \mathcal{X} be a Hilbert \mathcal{A} -module, $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a φ -map for an α -CP map $\varphi : \mathcal{A} \mapsto \mathcal{H}_1$ and (u', u) -covariant with respect to η . Then there exist a $\widehat{\pi_\varphi}$ -representation $\widehat{\pi_\mathcal{X}}$ of $\mathcal{G} \times_\eta \mathcal{X}$ on the Krein spaces (\mathcal{K}_1, J_3) and (\mathcal{K}_2, J_4) , and a unique $\tilde{\varphi}$ -map $\tilde{\Phi} : \mathcal{G} \times_\eta \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ for the $\tilde{\alpha}$ -CP map $\tilde{\varphi}$ such that*

$$\tilde{\Phi}(\widehat{x}) = W_\Phi^\# \widehat{\pi_\mathcal{X}}(\widehat{x}) V_\Phi = \int_{\mathcal{G}} \Phi(\widehat{x}(t)) u_t dt$$

for all $\widehat{x} \in C_c(\mathcal{G}, \mathcal{X})$.

Proof. Let $(\pi_\mathcal{X}, \pi_\varphi, V_\Phi, W_\Phi, v, v', (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_4))$ be the covariant KSGNS construction associated to Φ . Let $\widehat{x} \in C_c(\mathcal{G}, \mathcal{X})$. Since

$$\begin{aligned} \left| \left\langle \int_{\mathcal{G}} \pi_\mathcal{X}(\widehat{x}(t)) v_t \xi dt, \zeta \right\rangle \right| &= \int_{\mathcal{G}} |\langle \pi_\mathcal{X}(\widehat{x}(t)) v_t \xi, \zeta \rangle| dt \\ &\leq C(\widehat{x}) \int_{\mathcal{G}} \|\widehat{x}(t)\| \|\xi\| \|\zeta\| dt \\ &\leq C(\widehat{x}) \|\xi\| \|\zeta\| \|\widehat{x}\|_1 \end{aligned}$$

for all $\xi \in \mathcal{K}_1$, $\zeta \in \mathcal{K}_2$ and for some constant $C(\widehat{x}) > 0$, and also for any $x, y \in \mathcal{X}$, $a \in \mathcal{A}$ and $\xi, \zeta \in \mathcal{H}_1$, we obtain that $\pi_{\mathcal{X}}(\widehat{x}(t))v_t(a \otimes \xi) = \Phi(\widehat{x}(t)\beta_t^\eta(a))u_t\xi$ and

$$\begin{aligned}\Phi(\widehat{x}(t)\beta_t^\eta(a))^*\Phi(y) &= J_1\Phi(\widehat{x}(t)\beta_t^\eta(a))^\# \Phi(y) \\ &= J_1\varphi(\langle \widehat{x}(t)\beta_t^\eta(a), y \rangle) \\ &= J_1\varphi(\beta_t^\eta(a)^\# \alpha(\langle \widehat{x}(t), y \rangle)) \\ &= V_\Phi^\# \pi_\varphi(\beta_t^\eta(a))^* \pi_\varphi(\alpha(\langle \widehat{x}(t), y \rangle))\end{aligned}$$

and so we obtain

$$\begin{aligned}\langle \pi_{\mathcal{X}}(\widehat{x}(t))v_t(a \otimes \xi), \Phi(y)\zeta \rangle &= \langle u_t\xi, \Phi(\widehat{x}(t)\beta_t^\eta(a))^*\Phi(y)\zeta \rangle \\ &= \left\langle u_t J_1 \xi, J_1 V_\Phi^\# \pi_\varphi(\beta_t^\eta(a))^* \pi_\varphi(\alpha(\langle \widehat{x}(t), y \rangle)) \zeta \right\rangle \\ &= \langle a \otimes \xi, v_t^* \pi_\varphi(\alpha(\langle \widehat{x}(t), y \rangle)) \zeta \rangle.\end{aligned}$$

Therefore, there exists an element $\widehat{\pi_{\mathcal{X}}}(\widehat{x}) \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that

$$\widehat{\pi_{\mathcal{X}}}(\widehat{x}) = \int_{\mathcal{G}} \pi_{\mathcal{X}}(\widehat{x}(t)) v_t dt.$$

In this way, we obtain a map $\widehat{\pi_{\mathcal{X}}} : C_c(\mathcal{G}, \mathcal{X}) \rightarrow \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ being extended by continuity to a continuous map $\widehat{\pi_{\mathcal{X}}} : \mathcal{G} \times_{\eta} \mathcal{X} \rightarrow \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. Moreover, by applying (3.6), the covariance property of π_φ and the Fubini's theorem, we observe that

$$\begin{aligned}\widehat{\pi_\varphi}(\langle \widehat{x}, \widehat{y} \rangle) &= \int_{\mathcal{G}} \pi_\varphi(\langle \widehat{x}, \widehat{y} \rangle(t)) v_t dt = \int_{\mathcal{G}} \int_{\mathcal{G}} \pi_\varphi(\beta_{s^{-1}}^\eta(\langle \widehat{x}(s), \widehat{y}(st) \rangle)) v_t ds dt \\ &= \int_{\mathcal{G}} \int_{\mathcal{G}} v_{s^{-1}} \pi_\varphi(\langle \widehat{x}(s), \widehat{y}(st) \rangle) (v_{s^{-1}})^\# v_t ds dt \\ &= \int_{\mathcal{G}} \int_{\mathcal{G}} v_s^* \pi_{\mathcal{X}}(\widehat{x}(s))^\# \pi_{\mathcal{X}}(\widehat{y}(st)) v_{st} ds dt \\ &= \int_{\mathcal{G}} \int_{\mathcal{G}} J_3 v_s^* \pi_{\mathcal{X}}(\widehat{x}(s))^* J_4 \pi_{\mathcal{X}}(\widehat{y}(g)) v_g dg ds \\ &= J_3 \widehat{\pi_{\mathcal{X}}}(\widehat{x})^* J_4 \widehat{\pi_{\mathcal{X}}}(\widehat{y}) \\ &= \widehat{\pi_{\mathcal{X}}}(\widehat{x})^\# \widehat{\pi_{\mathcal{X}}}(\widehat{y})\end{aligned}$$

for all $\widehat{x}, \widehat{y} \in C_c(\mathcal{G}, \mathcal{X})$, $\widehat{\pi_{\mathcal{X}}}$ is a $\widehat{\pi_\varphi}$ -representation of $\mathcal{G} \times_{\eta} \mathcal{X}$ on the Krein spaces (\mathcal{K}_1, J_1) and (\mathcal{K}_2, J_2) . Consider the map $\widetilde{\Phi} : \mathcal{G} \times_{\eta} \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ defined by

$$\widetilde{\Phi}(z) = W_\Phi^\# \widehat{\pi_{\mathcal{X}}}(z) V_\Phi.$$

Then for all $\widehat{x} \in C_c(\mathcal{G}, \mathcal{X})$ we have

$$\begin{aligned}\widetilde{\Phi}(\widehat{x}) &= W_\Phi^\# \widehat{\pi_{\mathcal{X}}}(\widehat{x}) V_\Phi = \int_{\mathcal{G}} W_\Phi^\# \pi_{\mathcal{X}}(\widehat{x}(t)) v_t V_\Phi dt = \int_{\mathcal{G}} W_\Phi^\# \pi_{\mathcal{X}}(\widehat{x}(t)) V_\Phi u_t dt \\ &= \int_{\mathcal{G}} \Phi(\widehat{x}(t)) u_t dt.\end{aligned}$$

on the other hand, for any $z_1, z_2 \in \mathcal{G} \times_{\eta} \mathcal{X}$, we obtain that

$$\begin{aligned} \tilde{\Phi}(z_1)^{\#} \tilde{\Phi}(z_2) &= V_{\Phi}^{\#} \widehat{\pi_{\mathcal{X}}}(z_1)^{\#} W_{\Phi} W_{\Phi}^{\#} \widehat{\pi_{\mathcal{X}}}(z_2) V_{\Phi} = V_{\Phi}^{\#} \widehat{\pi_{\mathcal{X}}}(z_1)^{\#} \widehat{\pi_{\mathcal{X}}}(z_2) V_{\Phi} \\ &= V_{\Phi}^{\#} \widehat{\pi_{\varphi}}(\langle z_1, z_2 \rangle) V_{\Phi} = \int_{\mathcal{G}} V_{\Phi}^{\#} \pi_{\varphi}(\langle z_1, z_2 \rangle(t)) v_t V_{\Phi} dt \\ &= \int_{\mathcal{G}} V_{\Phi}^{\#} \pi_{\varphi}(\langle z_1, z_2 \rangle(t)) V_{\Phi} u_t dt = \int_{\mathcal{G}} \varphi(\langle z_1, z_2 \rangle(t)) u_t dt \\ &= \tilde{\varphi}(\langle z_1, z_2 \rangle) \end{aligned}$$

and so we conclude that $\tilde{\Phi}$ is a $\tilde{\varphi}$ -map. \square

Remark 3.7. Suppose that $(\pi_{\mathcal{X}}, \pi_{\varphi}, V_{\Phi}, W_{\Phi}, v, v', (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_3))$ is the minimal co-variant KSGNS construction associated to Φ . Then one can easily conclude that $(\widehat{\pi_{\mathcal{X}}}, V_{\Phi}, W_{\Phi}, (\mathcal{K}_1, J_3), (\mathcal{K}_2, J_4))$ is the minimal KSGNS construction associated to $\tilde{\Phi}$.

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